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INVARIANT VARIATIONAL PROBLEMS WITH BOUNDARY CONSTRAINTS

We construct equivariant version of Ljusternik – Schnirelman theory for the variational problems with boundary constraints. Our results can be proved in undeformation way, and no kind of constraint regularity is required.

1. Introduction.

There are many problems concerning the critical points of G-invariant functional $f(u) = \int\limits_{\Omega} a(x, u, \nabla u)\, dx$ with G-invariant operator constraint $F\left(u\big|_{\partial\Omega}\right) = 0$ which is defined on the boundary function values only. Here G is a compact subgroup of automorphisms group $\mathrm{Aut}\,(E)$ with unique fixed point — zero $0 \in E$. The space E is some closed subspace in a suitable Sobolev space of vector functions $W^1_{p,m}(\Omega) \ni u = \{u^1, \ldots, u^m\}$, Ω is a bounded domain in $\mathbb{R}^N$, gradient $\nabla u = \{\nabla u^1, \ldots, \nabla u^m\}$.

We shall indicate two examples of such problems. The first is some nonlinear analog of Steklov problem: scalar functions $u \in W^1_{2,1}(\Omega) \equiv W^1_2(\Omega)$, $f(u) = \int\limits_{\Omega} |\nabla u|^2\, dx$, $F(u) = \int\limits_{\partial\Omega} b(s, \gamma u)\, ds\, E \to \mathbb{R}$, γ is a boundary trace operator. In this case (under natural assumptions and if we will have carried out the change of $f(u)$ for $f(u) + F(u)$) all standard conditions of Ljusternik – Schnirelman theory are fulfilled. The critical points are the solutions of a problem

$$\text{(a)}\ \ \Delta u = 0, \quad \text{(b)}\ \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = \lambda b_u(s, u), \quad \text{(c)}\ \int\limits_{\partial\Omega} b(s, \gamma u)\, ds = 0, \tag{1.1}$$

(see details in [1], § V.1).

The second example is a domain mapping $u = \{u^1, \ldots, u^m\}\, \Omega \to \Omega_1$. Here the functional $f(u)$ and the space E would be so chosen as to ensure the mapping entrance to prescribed class. The constraint $F(\gamma u) = 0$ (here γ is a vector boundary trace operator) would be guarantee that $u\big|_{\partial\Omega} \in \partial\Omega_1$. If u is a mapping with bounded distortion, then last condition implies $u\big|_{\Omega} \in \Omega_1$ ([2], § 6.1).

At the same time, one is interested in investigation of not only critical set but of the more extensive set

$$B = \{u \in E :\ \ Lu = 0,\ \ F(u) = 0\}, \tag{1.2}$$

where $Lu = I^* f'(u)$ and $I \ \overset{\circ}{\to} W^1_{p,m} \hookrightarrow W^1_{p,m}$ is an embedding. In a regular case, when we define B, we do not consider specific boundary properties of solutions which are contained in boundary components of Euler – Lagrange equations (like (1.1.b)).

The proposed approach make possible to prove the existence in B the set of functions having totality or basis property. In addition in the first example there arise, instead of (1.1.b), some relations which lead to nonlinear analog of a basis for triangular operator representation. (The first example will be considered in details, the second one will not be considered since it require individual work.)

Our approach exploits the functional definition of G-genus (cf. [1], [3]) and does not exploit any deformation tools; it is closest to linear methods. Therefore no kind of constraint regularity [4], or quasiregularity [5], is required.

2. General definitions and propositions.

2.1. *G-genus of sets.* Let E be closed linear space, $\overset{\circ}{\to} W^1_p \subset E \subseteq W^1_p(\Omega)$, $1 < p < \infty$. Let G be compact topological group with (left) representation $T\, G \to \mathrm{Aut}\,(E)$, and also the map $\Theta\, G \times E \ni (g, u) \to T(g)u \in E$ be jointly continuous in the variables (g, u). (In the sequel we shall not distinguish g and Tg, and the notation "g" will being used in both cases.) Suppose zero is the unique fixed point for G-action in E.

Let μ be normed Haar measure in G and H be the space of all functionals $h(u)$, $u \in E$ with the following properties: (a) to be continuous; (b) to be bounded in bounded sets; (c) for all $u \in E$, $h \in H$ the means $mh(gu) \equiv \int\limits_G h(gu)\, d\mu(g) = 0$.

DEFINITION 1. *G-genus of the set $M \subset E$ is such an integer $Gr\, M$ that*

1) $Gr\, M = 0 \Longleftrightarrow M = \varnothing$;

2) $Gr\, M = k \Longleftrightarrow$ there are k functionals $h_i \in H$ (ample set of functionals) such that $\sum\limits_{i=1}^{k} h_i^2(u) > 0$ in M, and there are not $k-1$ functionals with the same inequality;

3) $Gr\, M = \infty$, iff there is not a finite ample set for M.

G-genus properties are analogous to Krasnosel'skiĭ genus ones [6]. In certain situations it is helpfully to have a similar definition for any subset $Q \subset H$, i. e. to use only $h_i \in Q$. It is involved in notation $QGr\, M$, and in this case the genus was studied in [1].

2.2. *Lower bounds for cardinality of B.* Let us list the required suppositions.

(i) Functional f is bounded below and coercive, i. e. $f(u) \to +\infty$ if $\|u\| \to \infty$.

(ii) Functional f is sequentially weakly lower semicontinuous, i. e. $u_i \rightharpoonup u_0 \Rightarrow f(u_0) \le \underline{\lim}\, f(u_i)$.

(iii) f is Gâteaux differentiable and $\langle f'(u), u\rangle > 0$ if $u \ne 0$.

(iv) $F\, E \to E_1$, where E_1 is Banach space. The set $\{F = 0\}$ is sequentially weakly closed and does not contain zero.

(v) $(\forall u \in E)(\forall v \in \overset{\circ}{\to} W^1_p(\Omega))\ (F(u) = 0) \Rightarrow (F(u + v) = 0)$.

(Further we shall assume, for a simplicity, that Ω is a smooth bounded domain.)

THEOREM 1. *There exist at least $k = Gr\,\{F = 0\}$ elements $e_i \in B$ such that* (1) $\langle f'(e_i), e_j\rangle = 0$, (2) $f(e_i) \le f(e_j)$, *if $i < j$.*

Proof. The existence of minimum $e_1 = \arg \min\limits_{F(u)=0} f(u)$ can be proved in a usual method [7], n° 1.10 that implies the conclusion of a Theorem for $k = 1$. Next, suppose $k > 1$.

For all $l \in E^*$, $u \in E$ the function $\langle l, gu\rangle$ on $g \in G$ is continuous and $m_l = m\langle l, gu\rangle$ define a bounded linear functional on l, if u is fixed. Thus, we defined some element $v \in E^{**} = E$ which will be denoted as $m(gu)$. Since $(\forall g_1 \in G)\ \langle l, g_1 m(gu)\rangle = \langle g_1^* l, m(gu)\rangle = m\langle g_1^* l, gu\rangle = m\langle l, g_1 gu\rangle = m\langle l, gu\rangle = \langle l, m(gu)\rangle$, it follows that $m(gu)$ is a fixed point, i. e. zero. Hence functionals $\langle l, u\rangle \in H$.

Let $h_1(u) = \langle f'(e_1), u\rangle$ and $M_1 = \{F = 0\} \cap \{h_1 = 0\}$. Since $k > 1$, the set M_1 is not empty, and its sequential weak completeness is obvious. Denote by e_2 the arg $\min_{u\in M_1} f(u)$. It is clear that $e_2 \neq e_1$: $h_1(e_2) = 0$ but $h_1(e_1) > 0$ by (iii), (iv). Due to condition (v) and the property $\forall v \in\!\stackrel{\circ}{\rightarrow} W_p^1(\Omega)$ $(h_1(u) = 0) \Rightarrow (h_1(u+v) = 0)$, we have (for the same v) $\langle f'(e_2), v\rangle = 0$, i. e. $e_2 \in B$.

If $k > 2$ then we may, analogously, to consider $M_2 = \{F = 0\} \cap \{h_1 = 0\} \cap \{h_2 = 0\}$, where $h_2(u) = \langle f'(e_2), u\rangle$, and on M_2 it can be obtained $e_3 = \arg\min f(u)$, etc. The number of the elements e_i is at least k, semiorthogonality follows from the conditions $h_1(e_2) = 0$, $h_2(e_3) = 0, \ldots$, the monotonicity of the values $f(e_i)$ follows from the insertions $M_1 \supset M_2 \supset \ldots$. □

REMARK 1. *Formally G-invariance of f, F is not used, but in fact G-invariance of F is required for $Gr\,\{F = 0\}$ calculations. As for invariance of f, it is not required here, in contrast to Ljusternik – Schnirelman theory, as G- equivariant deformations are not required. However, if we suppose that*

(vi) $(\forall u \in E, g \in G)\ f(gu) = f(u),\ F(gu) = F(u),$

then Theorem 1 can be sharpened. For an easy example, if $G = \{1, -1\}$, then there exist at least k pairs of elements $(e_i, -e_i)$ in B. For general groups the similar sharpening will be obtained in next section.

2.3. *The case $k = \infty$. Totality of the family $\{f'(e_i)\}$.* It is required here to strengthen conditions on f, and we bring new conditions not in most general but in easily verified forms. Further we shall assume that all $d_i(t)$ are continuous functions, $d_i(t) > 0$ if $t \neq 0$, $d_i(0) = 0$. The additional conditions on each $d_i(t)$, if they will be needed, will be specified.

So new conditions are the followings.

(vii) $f = f_0 + f_1$, $f_i \in C^1$, operator f_0' is uniformly monotone, i. e. $\langle f_0'(u) - f_0'(v), u - v\rangle \geq d_1(||u - v||)$; functional f_1 is strongly continuous, i. e. $u_i \rightharpoonup u_0 \Rightarrow f_1(u_i) \to f_1(u_0)$.

(viii) $|\langle f_0'(u), u\rangle| \leq d_2(f_0(u))$ where $d_2(t)$ is supposed monotone increasing; $f_0'(0) = 0$.

(ix) $f_0(u) \geq d_3(||u||)$, $d_3(t) \to +\infty$ at $t \to \infty$.

(x) $|f_1(u)| \leq d_4(f_0(u))f_0(u)$ where $\overline{\lim}_{t\to\infty} d_4(t) < 1$.

(xi) $Gr\,\{F = 0\} = \infty$.

THEOREM 2. *Let the conditions* (iv, v, vii–xi) *are fulfilled. Then:* (1) *the family $\{e_i\}$ from Theorem 1 is infinite;* (2) *$f(e_i) \to \infty$ at $i \to \infty$;* (3) *the family $\{f'(e_i)\}$ is total on $\{F = 0\}$, i. e. for all $u \in \{F = 0\}$ there exists e_j such that $\langle f'(e_j), u\rangle \neq 0$.*

If (vi) *is fulfilled, then we can claim:* (4) *there exist infinite number of G-orbits in the set B.*

Proof. Infinity of $\{e_i\}$ follows from (xi) and can be proved similarly Theorem 1. It follows from (vi) that the values of f on Ge_i are fixed, and if the second statement is proved, the number of different orbits would be infinite. Thus we need to prove the statements 2 and 3, but before its proving next lemma should be proved.

Lemma 1. *Let $\{u_n\}$ be such a sequence that $u_n \rightharpoonup u_0$ and $f_0(u_n) \to f_0(u_0) \overset{den}{=} \alpha$. Then $u_n \to u_0$.*

This lemma is analogous to well-known theorem: in uniformly convex Banach spaces if $u_n \rightharpoonup u_0$ and $||u_n|| \to ||u_0||$ then $u_n \to u_0$. In [8] this statement is considerably extended, but even the most general its formulation have not lemma 1 as a consequence: although global properties of f_0 are analogous to certain function $\varphi(||\cdot||)$, its local properties have not such similarities.

Proof of the Lemma. The conclusion of the lemma is obvious when $\alpha = 0$, therefore suppose $\alpha \neq 0$.

I. If $u_n \rightharpoonup u_0$, $\overline{\lim}\langle f_0'(u_n), u_n - u_0\rangle \leq 0$, then $u_n \to u_0$, it follows from (vii).

II. If $u_n \rightharpoonup u_0$, $f_0(u_n) \to f(u_0)$, $f_0\left(\dfrac{u_n + u_0}{2}\right) \to f_0(u_0)$, then $u_n \to u_0$. Indeed, for a suitable $\tau_n \in (0,1)$ we have

$$0 = \lim\left[f_0\left(\tfrac{u_n+u_0}{2} + \tfrac{u_n-u_0}{2}\right) - f_0\left(\tfrac{u_n+u_0}{2}\right)\right] = \lim\left\langle f'\left(\tfrac{u_n+u_0}{2} + \tau_n\tfrac{u_n-u_0}{2}\right), \tfrac{u_n-u_0}{2}\right\rangle = \lim\tfrac{1}{1+\tau_n}\left\langle f'\left(\tfrac{u_n+u_0}{2} + \tau_n\tfrac{u_n-u_0}{2}\right), \tfrac{u_n+u_0}{2} + \tau_n\tfrac{u_n-u_0}{2} - u_0\right\rangle.$$

Taking account of I, we obtain that $u_n \to u_0$.

III. For all $u \in \{f_0 = \alpha\}$ next inequality $\langle f_0'(u_0), u\rangle \leq \beta \overset{def}{=} \langle f_0'(u_0), u_0\rangle$ is fulfilled. Indeed, let $f_0(u) = \alpha$, $u \neq u_0$, $u_1 = \dfrac{u_0 + u}{2}$. Since f_0 is strictly convex, we have $f_0(u_1) = \alpha_1 < \alpha$. For $\varphi(t) = f_0((1-t)u_0 + tu_1)$ it will be $\varphi(0) = \alpha$, $\varphi(t) < \alpha(1-t) + \alpha_1 t = \alpha - t(\alpha - \alpha_1)$, i.e. the differentiable function $\varphi(t)$ is linearly decreasing near zero, hence $0 > \varphi'(0) = \langle f_0'(u_0), u_1 - u_0\rangle$.

IV. We will prove that, for all $\alpha_1 \in (0,\alpha)$ and $w \in \{f_0 = \alpha_1\}$, it will be true that $\langle f_0'(u_0), w\rangle \leq \beta - d_5(\alpha - \alpha_1)$. Here function $d_5(t)$ is increasing for $t > 0$, in addition to above conditions.

Since $\langle f_0'(u), u\rangle > 0$ for $u \neq 0$, $(f_0(tw) = \alpha) \Rightarrow (t > 1)$. Write out the lower bound for this t. $\left(\alpha - \alpha_1 = f_0(tw) - f_0(w) = \int_1^t \langle f_0'(sw), w\rangle\, ds \leq \int_1^t \langle f_0'(sw), sw\rangle\, ds \overset{(viii)}{\leq} \int_1^t d_2(f_0(sw))\, ds \leq d_2(\alpha)(t-1)\right) \Rightarrow \left(t \geq 1 + \dfrac{\alpha - \alpha_1}{d_2(\alpha)}\right)$.

From III $\langle f_0'(u_0), tw\rangle \leq \beta$, i.e.

$$\langle f_0'(u_0), w\rangle \leq \frac{\beta}{1 + (\alpha - \alpha_1)/d_2(\alpha)} = \beta - \frac{\beta(\alpha - \alpha_1)/d_2(\alpha)}{1 + (\alpha - \alpha_1)/d_2(\alpha)}.$$

The function $d_5(t) = \left|\dfrac{\beta t/d_2(\alpha)}{1 + t/d_2(\alpha)}\right|$ have prescribed form.

V. We shall complete the proof of lemma. Suppose $u_n \rightharpoonup u_0$, $f_0(u_n) \to f_0(u_0) = \alpha > 0$. If $u_n \not\to u_0$ then, for some subsequence, $f_0\left(\dfrac{u_{n_i} + u_0}{2}\right) \leq \alpha_1 < \alpha$ (to avoid the contradiction with II). Together with IV it implies $\left\langle f_0'(u_0), \frac{u_{n_i}+u_0}{2}\right\rangle \le \beta - d_5(\alpha-\alpha_1)$. On the other hand $\frac{1}{2}(\langle f_0'(u_0), u_{n_i}\rangle + \langle f_0'(u_0), u_0\rangle) \to \langle f_0'(u_0), u_0\rangle = \beta$. The contradiction prove the lemma. □

Let us continue the proof of the Theorem. Let $f(e_i) \le c < \infty$, then $\{e_i\}$ are bounded and let $e_{i_j} \rightharpoonup v$. Since (from (vii)) f is weakly lower semicontinuous, $f(v) \le \underline{\lim}\, f(e_{i_j})$. On the other hand v, as a weak limit of $\{e_{i_j}\}$, satisfies all constraints from Theorem 1, i. e. $v \in M_1, M_2, \ldots$. Since e_{i_j} are the minimums in M_{i_j}-problems, $f(v) \ge f(e_{i_j})$. So $f(e_{i_j}) \to f(v)$, i. e. $f_0(e_{i_j}) \to f_0(v)$ and (lemma 1) $e_{i_j} \to v$.

However, (iv) $\Rightarrow v \ne 0 \Rightarrow \langle f'(v), v\rangle \ge d_2(||v||) > 0$, contrary to the M_{i_j}-conditions: $\langle f'(e_{i_j}), v\rangle = 0$. The second conclusion is proved.

If there is $u_0 \in \{F = 0\}$ such that $(\forall n)\, \langle f'(e_n), u_0\rangle = 0$, then u_0 satisfies all the M_n-conditions, i. e. $(\forall n)\, f(u_0) \ge f(e_n)$. It is impossible, and we proved the third conclusion. □

Remark 2. *As it said after G-genus definition, we will have to reduce the class H to some $Q \subset H$, in certain cases. It is due to the (possible) fact: $HGr\,M$ may be small but $QGr\,M$ large. If $\langle f'(e_i), u\rangle \in Q$ then proofs above remain valid, otherwise we will change this form together with semiorthogonality relations.*

2.4. *Hilbert E, basis property.* In this section

(xii) $E = W^1_{2,m}(\Omega)$;

(xiii) $||\cdot||$ is Hilbert norm in E which is equivalent to standard one, $f(u) = \frac{1}{2}||u||^2$.

Some notations $E_{\partial\Omega}$ is the boundary traces space with the natural norm, generated by embedding. The space $E_0 = \operatorname{Ker}\gamma$, it is supposed an independed space and $I\; E_0 \to E$ is an embedding. The operator $J\; E^* \to E$ is dual, i. e. $Jf'\; E \to E$ is identity mapping. The space $(IE_0)_\perp$ is orthogonal complement to IE_0 in E and $(IE_0)^\perp$ is an annihilator of IE_0 in E^*. (In general, lower $\perp$ is orthogonal complement in an original space but upper is an annihilator in a dual one.)

Clearly, $(IE_0)_\perp = \operatorname{Ker} L$ and γ have a bounded right inverse $\gamma_r^{-1}\; E_{\partial\Omega} \to E$.

Suppose, in addition, that

(xiv) $F(u) = \Phi(\gamma u)$ where the operator $\Phi\; E_{\partial\Omega} \to E_1$ is strongly continuous, $\Phi(0) = 0$;

(xv) $\gamma\{F = 0\}$ is an absorbing set in $E_{\partial\Omega}$ in the following meaning $(\forall\, 0 \ne u \in E_{\partial\Omega})\; (\exists\lambda(u))\; \Phi(\lambda(u)u) = 0$; or (it is equivalent) the set $\{F = 0\}$ is an absorbing one in $\operatorname{Ker} L$.

(We shall use abstract notations E, E_0, etc, because, in spite of its concrete contents, the method of the proof is abstract, and the following theorem is applicable in other concrete situations.)

Theorem 3. *Let the conditions* (xi–xv) *are fulfilled. Then:*

- $\left\{\frac{e_i}{||e_i||}\right\}$ *is Hilbert basis in* $(IE_0)_\perp = \operatorname{Ker} L$;

- $\left\{ \frac{f'(e_i)}{||f'(e_i)||_*} \right\}$ *is Hilbert basis in* $(IE_0)^\perp$;
- $\left\{ \gamma \frac{e_i}{||e_i||} \right\}$ *is Hilbert basis in* $E_{\partial\Omega}$;
- $\left\{ (\gamma^*)^{-1} \frac{f'(e_i)}{||f'(e_i)||_*} \right\}$ *is Hilbert basis in* $E^*_{\partial\Omega}$.

Proof. 1) Since the conditions of the Theorem 2 (which ensure the totality of $\{f'(e_i)\}$ in $\{F = 0\}$) are fulfilled, the family $\{e_i\}$ is total in $(IE_0)_\perp$. Indeed, there exists, otherwise, such $v \in (IE_0)_\perp$, $v \neq 0$ that $(\forall i)\,(v, e_i) = 0$, i. e. v is orthogonal to the linear span $\mathcal{L}\{e_i\}$. Hence $v \in (\mathcal{L}\{f'(e_i)\})^\perp$. However some $\lambda(v)v \in \{F = 0\}$ and the family $\{f'(e_i)\}$ is total in $\{F = 0\}$.

Under given conditions the semiorthogonality is equivalent to orthogonality, hence, the family $\left\{ \frac{e_i}{||e_i||} \right\}$ is Hilbert basis in $(IE_0)_\perp$.

2) The isometry $f'\ (IE_0)_\perp \to E_0^\perp$ transform this basis into the basis $\left\{ \frac{f'(e_i)}{||f'(e_i)||_*} \right\}$ in $E_0^\perp$.

3+4) The operator $\gamma^*\ E^*_{\partial\Omega} \to E^*$, as is easily seen, is an isometry $E^*_{\partial\Omega}$ and $E_0^\perp \subset E^*$. Therefore $\left\| (\gamma^*)^{-1} \frac{f'(e_i)}{||f'(e_i)||_*} \right\|_{E^*_{\partial\Omega}} = 1$ and $\left\langle (\gamma^*)^{-1} \frac{f'(e_i)}{||f'(e_i)||_*}, \gamma \frac{e_j}{||e_j||} \right\rangle = \delta_{ij}$. The operator $\gamma J \gamma^*\ E^*_{\partial\Omega} \to E_{\partial\Omega}$ is a linear dual one, i. e. isometry $E^*_{\partial\Omega}$ onto $E_{\partial\Omega}$, therefore $\langle \gamma J \gamma^* v, u \rangle = (v, u)_{E^*_{\partial\Omega}}$.

Hence

$$\delta_{ij} = \left\langle (\gamma^*)^{-1} \frac{f'(e_i)}{||\cdot||}, \gamma \frac{e_j}{||\cdot||} \right\rangle =$$

$$= \left\langle (\gamma^*)^{-1} \frac{f'(e_i)}{||\cdot||}, \gamma J \gamma^* (\gamma^*)^{-1} \frac{f'(e_j)}{||\cdot||} \right\rangle = \left((\gamma^*)^{-1} \frac{f'(e_i)}{||\cdot||}, (\gamma^*)^{-1} \frac{f'(e_j)}{||\cdot||} \right)_{E^*_{\partial\Omega}}.$$

So $\left\{ (\gamma^*)^{-1} \frac{f'(e_i)}{||\cdot||} \right\}$ is orthonormal family in $E^*_{\partial\Omega}$, then in $E_{\partial\Omega}$ there will be orthonormal family $\left\{ \gamma \frac{e_i}{||\cdot||} \right\} = \gamma J \gamma^* \left\{ (\gamma^*)^{-1} \frac{f'(e_i)}{||\cdot||} \right\}$.

Since γ^* is an isometry $E^*_{\partial\Omega}$ onto $E_0^\perp$ and γ is an isometry $(IE_0)_\perp$ onto $E_{\partial\Omega}$, the families $\left\{ \gamma \frac{e_i}{||\cdot||} \right\}$ and $\left\{ (\gamma^*)^{-1} \frac{f'(e_i)}{||\cdot||} \right\}$ are total in the corresponding spaces, i. e. are basises. □

3. Example. Nonlinear Steklov problem.

This problem was already mentioned in introduction, see (1): when we defined B, the boundary part (1.b) of Euler–Lagrange system was not considered. However our specific method of $\{e_i\}$ construction bring us new boundary equations and new function family, which will be of interest. As the author would be about to show this effect only, we shall consider a simplest variant. Let Ω be bounded smooth domain in $\mathbb{R}^2$, ν be the outer normal, $E = W_2^1(\Omega)$, $f(u) = \frac{1}{2}\int\limits_{\Omega}(|\nabla u|^2 + u^2)dx$, $x = (x^1, x^2)$, $F(u) = \frac{1}{4}\int\limits_{\partial\Omega} u^4\, ds - 1$, $Gu = \{u, -u\}$.

All the conditions from the section 2.4 are valid, and we need only to explain what kind of functions $\{e_i\}$ are with regard to their new natural boundary conditions.

Let us write out the Euler – Lagrange equations (it is enough for e_1, e_2, e_3):

$$\lambda_{1,1}f'(e_1) + \mu_1 F'(e_1) = 0, \quad \lambda_{2,1}f'(e_1) + \lambda_{2,2}f'(e_2) + \mu_2 F'(e_2) = 0,$$
$$\lambda_{3,1}f'(e_1) + \lambda_{3,2}f'(e_2) + \lambda_{3,3}f'(e_3) + \mu_3 F'(e_3) = 0.$$

Note that, for example, the first equation is the generalized form of the equation $-\Delta e_1 + e_1 = 0$ and boundary condition $\lambda_{1,1}\frac{\partial e_1}{\partial \nu} + \mu_1 e_1^3\Big|_{\partial\Omega} = 0$. However it is more convenient to leave the system in abstract form.

It folows, from the orthogonality of $\{e_i\}$, that all $\mu_i \neq 0$. Hence we can, without loss of generality, to rewrite the system in the next form, with retaining the old notations for new λ.

$$F'(e_1) = \lambda_{1,1}f'(e_1)\,, \tag{3.1}$$
$$F'(e_2) = \lambda_{2,1}f'(e_1) + \lambda_{2,2}f'(e_2) \tag{3.2}$$
$$F'(e_3) = \lambda_{3,1}f'(e_1) + \lambda_{3,2}f'(e_2) + \lambda_{3,3}f'(e_2)\,. \tag{3.3}$$

After the multiplications of (3.1–3.3) on e_1 or e_2, e_3 and integral $\langle\cdot,\cdot\rangle$ evaluations we obtain ($\times e_1$:)

$$\lambda_{1,1} = \frac{\langle F'(e_1), e_1\rangle}{\langle f'(e_1), e_1\rangle}\,,$$

$$\lambda_{2,1} = \frac{\langle F'(e_2), e_1\rangle}{\langle f'(e_1), e_1\rangle} = \frac{\langle F'(e_2), e_1\rangle - \langle F'(e_1), e_2\rangle}{\langle f'(e_1), e_1\rangle}\,,$$

$$(\text{as } \langle F'(e_1), e_2\rangle = \lambda_{1,1}\langle f'(e_1), e_2\rangle = 0)\,,$$

$$\lambda_{3,1} = \frac{\langle F'(e_3), e_1\rangle - \langle F'(e_1), e_3\rangle}{\langle f'(e_1), e_1\rangle}\,.$$

($\times e_2$:)

$$\lambda_{2,2} = \frac{\langle F'(e_2), e_2\rangle}{\langle f'(e_2), e_2\rangle}\,,$$

$$\lambda_{3,2} = \frac{\langle F'(e_3), e_2\rangle - \langle F'(e_2), e_3\rangle}{\langle f'(e_2), e_2\rangle}\,,$$

$$(\text{as } \langle F'(e_2), e_3\rangle = \lambda_{2,1}\langle f'(e_1), e_3\rangle + \lambda_{2,2}\langle f'(e_2), e_3\rangle = 0)\,.$$

($\times e_3$:)

$$\lambda_{3,3} = \frac{\langle F'(e_3), e_3\rangle}{\langle f'(e_3), e_3\rangle}\,.$$

Let us discuss these formulas. For a completely continuous operator its (possible) nondiagonality is a consequence of normality absence. Therefore triangular or Jordan representations are the tools for nonsymmetric operators analysis. Nonlinear potential operators inherit some properties both from linear symmetric and from linear nonsymmetric operators. Symmetry of derivative is the first kind example, but the inequality $\langle Au, v\rangle \neq \langle Av, u\rangle$ (whose cause is just nonlinearity) is the second kind example.

Nonsymmetry, in turn, implies the triangular representation (3.i). Nondiagonal numbers $\lambda_{i,j}$ have normalizing denominators $\langle f'(e_j), e_j\rangle$ and numerators $\langle F'(e_i), e_j\rangle - \langle F'(e_j), e_i\rangle$ which registered just "nonsymmetry from nonlinearity" for operator F'.

If a function $u \in E_{\partial\Omega}$ have an expansion $u = (a_1 e_1^3 + a_2 e_2^3 + \ldots)^{1/3}$, then (for our example) $F'(u) = \sum a_i F'(e_i)$. After that we can exploit (3.i), and hence we can extend the triangular form of F' onto that functions.

The question, what kind of functions admits that epansion, falls outside the limits of this work subject.

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